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THE PROBLEM OF POWER-CONSUMPTION-OPTIMAL REORIENTATION WITH SIMULTANEOUS RETARDATION OF A SPHERICALLY SYMMETRIC BODY WITH AN UNSPECIFIED TIME†

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Some special features of the solutions of the problem of the optimal control of the spatial reorientation and simultaneous total retardation of the initial rotation of an absolutely solid spherically symmetric body in the case of an unspecified time are studied. The principal moment of the external applied forces serves as the control. The quality of the control process is estimated by an integral functional which characterizes the overall power consumption required to accomplish the manoeuvre. In a special case, this functional has the form of the well-known integral-quadratic functional. It is established that the problem of controlling the reorientation and simultaneous retardation of a rigid body with an unspecified time in the class of measurable controls, which is optimal with respect to power consumption, has no solutions for almost any initial conditions. One of the possible minimizing sequences is explicitly constructed. It is shown that the smallest values of the objective functionals in the problem of reorientation with retardation and in the problem of the total retardation of the initial rotation are identical. In particular, zero minimum power consumptions corresponds to the reorientation of a spherically symmetric body from a position of rest into a position of rest if the time at which the process terminates is not fixed. The uniqueness of the solution of the problem of optimal retardation is proved when an additional assumption is made concerning rigorous normalization.

The geometrical characteristic of the optimal turns of a symmetric body in a position of rest were investigated in [1, 2] and certain properties of Hamiltonian systems arising as a result of the application of the formalism of the maximum principle were considered. At the present time, there are many papers which analyse problems of the optimal control of angular motion. However, due to the substantial non-linearity of these problems, there are practically no results relating to the problem of the existence of solutions and proofs of their optimality. In this paper, the conditions for the existence of a solution in the problem of reorientation with simultaneous retardation of the initial rotation are obtained using time deformation transforms.

1. FORMULATION OF THE PROBLEM

We shall use elements of the matrix $A \in SO(3)$ of the direction cosines as the kinematric parameters of the angular motion. This matrix describes the change in the mutual position of the local and inertial system of coordinates with a common origin at the centre of mass [3, 4]. We shall write the equation in projections onto the axes of the local system of coordinates. Suppose $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T \in \mathbb{R}^3$ is the angular velocity vector, $S(\boldsymbol{\omega})$ is the skew-symmetric matrix

$$S(\boldsymbol{\omega}) = \begin{Vmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{Vmatrix}$$

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and $\mathbf{u} \in \mathbb{R}^3$ is the external control moment. A spatial manoeuvre of reorientation with simultaneous retardation of the initial rotation of a spherically symmetric body with unit inertial tensor is considered. This is described by the boundary-value problem

$$\begin{aligned} &A(0) = B, \quad \dot{A} = -S(\boldsymbol{\omega})A, \quad A(T) = C \\ &\boldsymbol{\omega}(0) = \mathbf{v}, \quad \dot{\boldsymbol{\omega}} = \mathbf{u} \quad \text{almost everywhere} \quad t \in [0, T], \quad \boldsymbol{\omega}(T) = 0 \end{aligned} \tag{1.1}$$

The controls $\mathbf{u}(t)$ will be chosen from the class of measurable functions of time. The time, T, at which the process terminates is not specified.

We will choose the four-parameter family of functions of the form

$$J = J(T, \mathbf{u}, \boldsymbol{\omega}) = a\|\mathbf{u}\|^{p_2}_{L^3_{p_1}([0,T]),\gamma} + b\|\boldsymbol{\omega}\|^{r_2}_{L^3_{r_1}([0,T]),\sigma}; \quad a > 0, b > 0 \tag{1.2}$$

as the criterion which characterizes the value of the overall power consumption for the execution of the manoeuvre. The norm of the three-dimensional function in the space $L^3_p([0, T])$ is introduced in the usual manner

$$\|\mathbf{u}\|_{L^3_p([0,T]),\gamma} = \left(\int_{[0,T]} |\mathbf{u}(t)|^p_\gamma d\mu\right)^{1/p}, \quad 1 \le p < \infty$$

where μ is the Lebesgue measure and $|\cdot|_\gamma$ and $|\cdot|_\sigma$ are vector norms in $\mathbb{R}^3$. The purpose of the control is to minimize the power consumption

$$J \to \inf \tag{1.3}$$

where the lower part is sought over all possible admissible trajectories, controls and termination times.

The numbers p_1, p_2, r_1, r_2 are the parameters. The ranges of possible values of these numbers are taken to be as follows:

$$1 \le p_1 < \infty, \quad p_2 > 0, \quad 1 < r_1 < \infty, \quad r_2 > 0 \tag{1.4}$$

This choice of a family of functions is explained by the fact that all of them, to a certain extent, characterize power consumptions, and the choice of the corresponding values depends to a large extent on the investigator and the specific slave mechanisms. The version $p_1 = p_2 = r_1 = r_2 = 2$, which corresponds to an integral-quadratic problem, is the most common. If, however, rocket motors serve as the slave mechanisms, then the choice of the values $p_1 = p_2 = 1$, $r_1 = r_2 = 2$ is more convenient since, in this case, the first term characterizes the overall consumption of the working body and the second term characterizes the overall kinetic energy.

We will now introduce definitions which will be required later. We call a process which is admissible in boundary-value problem (1.1)–(1.3) a quadruplet

$$(T, \mathbf{u}(t), \boldsymbol{\omega}(t), A(t); t \in [0, T]) \in [0, \infty) \times L^3_{p_1}([0, T]) \times \mathrm{AC}^3([0, T]) \times \mathrm{AC}^{3\times 3}([0, T])$$

the elements of which satisfy the differential equations, the boundary conditions and the condition $J(T, \mathbf{u}, \boldsymbol{\omega}) < \infty$. For given $B, C \in \mathrm{SO}(3)$ and $\mathbf{v} \in \mathbb{R}^3$, we will denote the set of all processes which are permissible in problem (1.1)–(1.3) by $\mathscr{X}(B, C, \mathbf{v})$. Problem (1.1)–(1.3) therefore consists of finding

$$\inf_{\mathscr{X}(B,C,\mathbf{v})} J(T, \mathbf{u}, \boldsymbol{\omega}) \tag{1.5}$$

By virtue of the non-obvious construction of the set $\mathscr{X}(B, C, \mathbf{v})$, it is not possible to use the classical results (a modification of Weierstrass' theorem). Arguments will therefore now be put forward which enable one to draw certain conclusions concerning the special features of problem (1.1)–(1.3).

2. MINIMIZING SEQUENCES IN THE PROBLEM OF REORIENTATION

It is obvious that $\mathscr{X}(B, C, \mathbf{v}) \neq \varnothing$ for any set of boundary conditions. In non-linear problem (1.1)–(1.3), it is possible to construct a solution using space – time decomposition. In fact, we shall show that a minimizing sequence of processes can be constructed as a sequence of two manoeuvres, one following the other: total retardation and, then, reorientation from a position of rest into a position of rest. The optimality of the geometry of the well-known solution, admissible in boundary-value problem (1.1), in the form of two successive planar turns will thus be confirmed.

We will now consider the problem of total retardation in an unspecified time

$$\boldsymbol{\omega}(0) = \mathbf{v}, \quad \dot{\boldsymbol{\omega}} = \mathbf{u} \quad \text{almost everywhere} \quad t \in [0, T], \quad \boldsymbol{\omega}(T) = 0 \tag{2.1}$$

$$J(T, \mathbf{u}, \boldsymbol{\omega}) \to \inf \tag{2.2}$$

By a process which is permissible in problem (2.1), (2.2), we shall mean the triple

$$(T, \mathbf{u}(t), \boldsymbol{\omega}(t); T \in [0, T]) \in [0, \infty) \times L^3_{p_1}([0, T]) \times \mathrm{AC}^3([0, T])$$

the elements of which satisfy the differential equation, the conditions on the right-hand and left-hand ends and for which $J(T, \mathbf{u}, \boldsymbol{\omega}) < \infty$ almost everywhere in $[0, T]$. For a given $\mathbf{v} \in \mathbb{R}^3$, we will denote the set of all processes which are admissible in boundary-value problem (2.1) by $\mathcal{Y}(\mathbf{v})$. In this case, the optimal control problem (2.1), (2.2) consists of finding $\inf_{\mathcal{Y}(\mathbf{v})} J(T, \mathbf{u}, \boldsymbol{\omega})$.

We note again that $\mathcal{Y}(\mathbf{v}) \neq \varnothing$ and introduce the notation

$$\hat{J}(\mathbf{v}) = \inf_{\mathcal{Y}(\mathbf{v})} J(T, \mathbf{u}, \boldsymbol{\omega}) < \infty$$

If

$$(T, \mathbf{u}(t), \boldsymbol{\omega}(t), A(t); t \in [0, T])$$

is an arbitrary admissible process in boundary-value problem (1.1)–(1.3), then the process

$$(T, \mathbf{u}(t), \boldsymbol{\omega}(t); t \in [0, T])$$

corresponding to it is admissible in problem (2.1), (2.2). The inequality

$$\inf_{\mathcal{X}(B, C, \mathbf{v})} J(T, \mathbf{u}, \boldsymbol{\omega}) \geq \inf_{\mathcal{Y}(\mathbf{v})} J(T, \mathbf{u}, \boldsymbol{\omega}) \tag{2.3}$$

therefore holds. It will be proved later that it is actually the equality which holds.

We will now first consider the problem of reorientation from a position of rest into a position of rest with an unspecified completion time

$$\begin{aligned} &A(0) = B, \quad \dot{A} = -S(\boldsymbol{\omega})A, \quad A(T) = C \\ &\boldsymbol{\omega}(0) = 0, \quad \dot{\boldsymbol{\omega}} = \mathbf{u} \quad \text{almost everywhere} \quad t \in [0, T], \quad \boldsymbol{\omega}(T) = 0 \\ &J(T, \mathbf{u}, \boldsymbol{\omega}) \to \inf \end{aligned} \tag{2.4}$$

We will show that, for any choice of $B, C \in \mathrm{SO}(3)$, the equality

$$\inf_{\mathcal{X}(B, C, 0)} J(T, \mathbf{u}, \boldsymbol{\omega}) = 0 \tag{2.5}$$

holds, for which we construct the corresponding minimizing sequence in the following manner.

It is clear that $\mathcal{X}(B, C, 0) \neq \varnothing$, and we therefore choose any admissible process

$$(T, \mathbf{u}(t), \boldsymbol{\omega}(t), A(t); t \in [0, T])$$

from $\mathcal{X}(B, C, 0)$. Now, for each natural k, we determine a new process

$$(T_k, \mathbf{u}_k(t), \boldsymbol{\omega}_k(t), A_k(t); t \in [0, T_k])$$

using the formulae

$$A_k(kt) = A_1(t), \quad \boldsymbol{\omega}_k(kt) = k^{-1}\boldsymbol{\omega}_1(t); \quad \mathbf{u}_k(kt) = k^{-2}\mathbf{u}_1(t) \quad \text{almost everywhere} \quad t \in [0, T_1]$$

where

$$A_1(t) = A(t), \quad \boldsymbol{\omega}_1(t) = \boldsymbol{\omega}(t), \quad \mathbf{u}_1(t) = \mathbf{u}(t), \quad T_1 = T$$

These formulae take a more convenient form if we change to the new variable $\tau = kt$ in them

$$\begin{aligned} &A_k(\tau) = A_1(k^{-1}\tau), \quad \boldsymbol{\omega}_k(\tau) = k^{-1}\boldsymbol{\omega}_1(k^{-1}\tau) \\ &\mathbf{u}_k(\tau) = k^{-2}\mathbf{u}_1(k^{-1}\tau) \ \text{ almost everywhere } \quad T_k = kT_1, \quad \tau \in [0, T_k] \end{aligned} \tag{2.6}$$

These relations describe the simplest version of time deformation, that is, time dilatation.

We will now show that a process, constructed using formulae (2.6), is admissible in boundary-value problem (2.4). Actually, the following conditions on the left-hand and right-hand ends are satisfied for each natural k

$$A_k(0) = A_1(0) = B, \quad A_k(T_k) = A_1(T_1) = C$$

$$\boldsymbol{\omega}_k(0) = k^{-1}\boldsymbol{\omega}_1(0) = 0, \quad \boldsymbol{\omega}_k(T_k) = k^{-1}\boldsymbol{\omega}_1(T_1) = 0$$

Direct calculations show that the differential equations of problem (2.4)

$$\dot{A}_k(\tau) = \frac{dA_k(\tau)}{d\tau} = \frac{dA_1(k^{-1}\tau)}{d\tau} = -S(k^{-1}\boldsymbol{\omega}_1(k^{-1}\tau))A_1(k^{-1}\tau) = -S(\boldsymbol{\omega}_k(\tau))A_k(\tau)$$

$$\dot{\boldsymbol{\omega}}_k(\tau) = \frac{d\boldsymbol{\omega}_k(\tau)}{d\tau} = k^{-1}\frac{d\boldsymbol{\omega}_1(k^{-1}\tau)}{d\tau} = k^{-2}\mathbf{u}_1(k^{-1}\tau) = \mathbf{u}_k(\tau)$$

also hold for (almost all) $\tau \in [0, T_k]$.

We will now estimate the corresponding values of the functionals. For each natural k, we have

$$J(T_k, \mathbf{u}_k, \boldsymbol{\omega}_k) = a\left(\int\limits_{[0, T_k]} |\mathbf{u}_k(\tau)|_\gamma^{p_1} d\mu\right)^{\frac{p_2}{p_1}} + b\left(\int\limits_{[0, T_k]} |\boldsymbol{\omega}_k(\tau)|_\sigma^{r_1} d\mu\right)^{\frac{r_2}{r_1}} =$$

$$= a\left(\int\limits_{[0, kT_1]} |k^{-2}\mathbf{u}_1(k^{-1}\tau)|_\gamma^{p_1} d\mu\right)^{\frac{p_2}{p_1}} + b\left(\int\limits_{[0, kT_1]} |k^{-1}\boldsymbol{\omega}_1(k^{-1}\tau)|_\sigma^{r_1} d\mu\right)^{\frac{r_2}{r_1}} =$$

$$= ak^{\frac{p_2}{p_1}(1-2p_1)}\left(\int\limits_{[0, T]} |\mathbf{u}(s)|_\gamma^{p_1} d\mu\right)^{\frac{p_2}{p_1}} + bk^{\frac{r_2}{r_1}(1-r_1)}\left(\int\limits_{[0, T]} |\boldsymbol{\omega}(s)|_\sigma^{r_1} d\mu\right)^{\frac{r_2}{r_1}} =$$

$$= ak^{\frac{p_2}{p_1}(1-2p_1)} \|\mathbf{u}\|^{p_2}_{L^3_{p_1}([0, T]), \gamma} + bk^{\frac{r_2}{r_1}(1-r_1)} \|\boldsymbol{\omega}\|^{r_2}_{L^3_{r_1}([0, T]), \sigma}$$

Since the process

$$(T, \mathbf{u}(t), \boldsymbol{\omega}(t), A(t); t \in [0, T])$$

was selected from $\mathscr{X}(B, C, 0)$, then $J(T, \mathbf{u}, \boldsymbol{\omega}) < \infty$. Consequently

$$\|\mathbf{u}\|^{p_2}_{L^3_{p_1}([0, T]), \gamma} < \infty, \quad \|\boldsymbol{\omega}\|^{r_2}_{L^3_{r_1}([0, T]), \sigma} < \infty$$

and therefore $J(T_k, \mathbf{u}_k, \boldsymbol{\omega}_k) < \infty$. Hence

$$(T_k, \mathbf{u}_k(t), \boldsymbol{\omega}_k(t), A_k(t); t \in [0, T_k]) \in \mathscr{X}(B, C, 0)$$

for all natural k. From conditions (1.4), we get

$$\frac{p_2}{p_1}(1-2p_1)<0, \quad \frac{r_2}{r_1}(1-r_1)<0$$

Consequently

$$J(T_k, \mathbf{u}_k, \boldsymbol{\omega}_k) \to 0 \quad \text{when} \quad k \to \infty$$

which is equivalent to equality (2.5).

Note that a minimizing sequence is constructed using formulae (2.6) for any initial admissible process from $\mathcal{X}(B, C, 0)$.

We will now construct the minimizing sequence in the initial problem (1.1)–(1.3). Suppose that

$$\{(t_n, \mathbf{u}_n(t), \boldsymbol{\omega}_n(t); t \in [0, t_n])\} \in \mathcal{Y}(\mathbf{v}), \quad n = 1, 2, \ldots$$

is an arbitrary minimizing sequence of admissible processes in problem (2.1), (2.2), that is

$$J(t_n, \mathbf{u}_n, \boldsymbol{\omega}_n) \to \hat{J}(\mathbf{v}) \quad \text{when} \quad n \to \infty \tag{2.7}$$

For each natural m, we find a number n_m and, also, the corresponding process

$$(t_{n_m}, \mathbf{u}_{n_m}(t), \boldsymbol{\omega}_{n_m}(t); t \in [0, t_{n_m}]) \in \mathcal{Y}(\mathbf{v})$$

for which the inequality

$$\left|\hat{J}(\mathbf{v}) - J(t_{n_m}, \mathbf{u}_{n_m}, \boldsymbol{\omega}_{n_m})\right| < \frac{1}{2m}$$

holds.

This is possible by virtue of condition (2.7).

Suppose $A_{n_m}(t)$ is the solution of the Cauchy problem

$$A_{n_m}(0) = B, \quad \dot{A}_{n_m} = -S(\boldsymbol{\omega}_{n_m})A_{n_m}, \quad t \in [0, t_{n_m}]$$

which corresponds to the chosen process

$$(t_{n_m}, \mathbf{u}_{n_m}(t), \boldsymbol{\omega}_{n_m}(t); t \in [0, t_{n_m}])$$

For m and n_m, we form the process

$$(t_{n_m}, \mathbf{u}_{n_m}(t), \boldsymbol{\omega}_{n_m}(t), A_{n_m}(t); t \in [0, t_{n_m}])$$

and using it, together with formulae (2.6), we construct the sequence of processes

$$\{(t_{n_m,k}, \mathbf{u}_{n_m,k}(t), \boldsymbol{\omega}_{n_m,k}(t), A_{n_m,k}(t); t \in [0, t_{n_m,k}])\} \subset \mathcal{X}(C_{n_m}, C, \mathbf{v}), \quad k = 1, 2, \ldots$$

where $C_{n_m} = A_{n_m}(t_{n_m})$, that is, for each natural k, we put

$$t_{n_m,k} = kt_{n_m}, \quad \mathbf{u}_{n_m,k}(\tau) = k^{-2}\mathbf{u}_{n_m}(k^{-1}\tau) \quad \text{almost everywhere}$$

$$\boldsymbol{\omega}_{n_m,k}(\tau) = k^{-1}\boldsymbol{\omega}_{n_m}(k^{-1}\tau), \quad A_{n_m,k}(\tau) = A_{n_m}(k^{-1}\tau), \quad \tau \in [0, t_{n_m,k}]$$

For fixed n and n_m, we choose a number k_m and, also, the corresponding process

$$(t_{n_m,k_m}, \mathbf{u}_{n_m,k_m}(t), \boldsymbol{\omega}_{n_m,k_m}(t), A_{n_m,k_m}(t); t \in [0, t_{n_m,k_m}]) \in \mathcal{X}(C_{n_m}, C, \mathbf{v})$$

for which the inequality

$$J(t_{n_m,k_m}, \mathbf{u}_{n_m,k_m}, \boldsymbol{\omega}_{n_m,k_m}) < \frac{1}{2m}$$

holds. This is possible by virtue of equality (2.5).

For each natural m, we now determine the functions using the formulae

$$\mathbf{u}_m^*(t) = \begin{cases} \mathbf{u}_{n_m}(t) & \text{almost everywhere} \quad t \in [0, t_{n_m}] \\ \mathbf{u}_{n_m, k_m}(t - t_{n_m}) & \text{almost everywhere} \quad t \in (t_{n_m}, t_m^*] \end{cases}$$

$$\boldsymbol{\omega}_m^*(t) = \begin{cases} \boldsymbol{\omega}_{n_m}(t) & t \in [0, t_{n_m}] \\ \boldsymbol{\omega}_{n_m, k_m}(t - t_{n_m}) & t \in (t_{n_m}, t_m^*] \end{cases}$$

$$A_m^*(t) = \begin{cases} A_{n_m}(t) & t \in [0, t_{n_m}] \\ A_{n_m, k_m}(t - t_{n_m}) & t \in (t_{n_m}, t_m^*] \end{cases}$$

$$t_m^* = t_{n_m} + t_{n_m, k_m}$$

It can be verified directly that the inclusion

$$(t_m^*, \mathbf{u}_m^*(t), \boldsymbol{\omega}_m^*(t), A_m^*(t); t \in [0, t_m^*]) \in \mathscr{X}(B, C, \mathbf{v})$$

holds for each $m = 1, 2, \ldots$.

We will now show that the sequence

$$\{(t_m^*, \mathbf{u}_m^*(t), \boldsymbol{\omega}_m^*(t), A_m^*(t); t \in [0, t_m^*])\}, \quad m = 1, 2, \ldots$$

is the minimizing sequence in the initial problem (1.1)–(1.3). In fact, we have

$$\left|\hat{J}(\mathbf{v}) - J(t_m^*, \mathbf{u}_m^*, \boldsymbol{\omega}_m^*)\right| = \left|\hat{J}(\mathbf{v}) - \left\{a\|\mathbf{u}_m^*\|^{p_2}_{L^3_{p_1}([0, t_{n_m}]), \gamma} + b\|\boldsymbol{\omega}_m^*\|^{r_2}_{L^3_{r_1}([0, t_{n_m}]), \sigma}\right\} -\right.$$

$$\left. - \left\{a\|\mathbf{u}_m^*\|^{p_2}_{L^3_{p_1}([t_{n_m}, t_m^*]), \gamma} + b\|\boldsymbol{\omega}_m^*\|^{r_2}_{L^3_{r_1}([t_{n_m}, t_m^*]), \sigma}\right\}\right| \le$$

$$\le \left|\hat{J}(\mathbf{v}) - \left\{a\|\mathbf{u}_m^*\|^{p_2}_{L^3_{p_1}([0, t_{n_m}]), \gamma} + b\|\boldsymbol{\omega}_m^*\|^{r_2}_{L^3_{r_1}([0, t_{n_m}]), \sigma}\right\}\right| +$$

$$+ \left|\left\{a\|\mathbf{u}_m^*\|^{p_2}_{L^3_{p_1}([t_{n_m}, t_m^*]), \gamma} + b\|\boldsymbol{\omega}_m^*\|^{r_2}_{L^3_{r_1}([t_{n_m}, t_m^*]), \sigma}\right\}\right| =$$

$$= \left|\hat{J}(\mathbf{v}) - J(t_{n_m}, \mathbf{u}_{n_m}, \boldsymbol{\omega}_{n_m})\right| + J(t_{n_m, k_m}, \mathbf{u}_{n_m, k_m}, \boldsymbol{\omega}_{n_m, k_m}) < \frac{1}{m}$$

Hence

$$J(t_m^*, \mathbf{u}_m^*, \boldsymbol{\omega}_m^*) \to \hat{J}(\mathbf{v}) \quad \text{as} \quad m \to \infty$$

and, from inequality (2.3), we now obtain the equality

$$\inf_{\mathscr{X}(B, C, \mathbf{v})} J(T, \mathbf{u}, \boldsymbol{\omega}) = \inf_{\mathscr{Y}(\mathbf{v})} J(T, \mathbf{u}, \boldsymbol{\omega}) \tag{2.8}$$

for arbitrary $B, C \in \mathrm{SO}(3)$ and $\mathbf{v} \in \mathbb{R}^3$.

The minimizing sequences in problem (1.1)–(1.3) of reorientation with simultaneous retardation can be constructed in a different way. For example, another type of manoeuvre can be chosen. However, it can be shown that all the minimizing sequences in a problem with an unspecified completion time possess a single common property: the completion time of the control processes tends to infinity.

3. THE LOWER LIMIT IN THE PROBLEM OF RETARDATION

If $\mathbf{v} = 0$, the solution of the problem of optimal retardation (2.1), (2.2) is obvious: $T = 0$, and therefore

$$\inf_{\mathcal{Y}(\mathbf{v})} J(T, \mathbf{u}, \boldsymbol{\omega}) = 0 \tag{3.1}$$

If $\mathbf{v} \neq 0$, then, for any choice of the parameters of the functional $a > 0$, $b > 0$, $p_1, r_1 \in [1, \infty)$ and $p_2, r_2 > 0$, the inequality

$$\inf_{\mathcal{Y}(0)} J(T, \mathbf{u}, \boldsymbol{\omega}) > 0 \tag{3.2}$$

always holds.

We shall argue that the opposite is true and assume that there is a sequence of admissible processes

$$\{(t_n, \mathbf{u}_n(t), \boldsymbol{\omega}_n(t);\ t \in [0, t_n])\} \subset \mathcal{Y}(\mathbf{v})$$

which are solutions of the boundary-value problem

$$\boldsymbol{\omega}_n(0) = \mathbf{v}, \quad \dot{\boldsymbol{\omega}} = \mathbf{u}_n \text{ almost everywhere } t \in [0, t_n], \quad \boldsymbol{\omega}_n(t_n) = 0 \tag{3.3}$$

by construction and, simultaneously

$$J(t_n, \mathbf{u}_n, \boldsymbol{\omega}_n) = a\left(\int_{[0, t_n]} |\mathbf{u}_n(t)|_\gamma^{p_1} d\mu\right)^{p_2/p_1} + b\left(\int_{[0, t_n]} |\boldsymbol{\omega}_n(t)|_\sigma^{r_1} d\mu\right)^{r_2/r_1} \to 0 \tag{3.4}$$

when $n \to \infty$.

If $\mathbf{v} = (\upsilon^1, \upsilon^2, \upsilon^3)^T \in \mathbb{R}^3$ and $\mathbf{u}_n(t) = (u_n^1(t), u_n^2(t), u_n^3(t))^T$, then, from Eqs (3.3), we obtain

$$-\upsilon^i = \int_{[0, t_n]} u_n^i(t) d\mu, \quad i = 1, 2, 3,$$

whence

$$|\upsilon^i| = \left|\int_{[0, t_n]} u_n^i(t) d\mu\right| \leq \int_{[0, t_n]} |u_n^i(t)| d\mu, \quad i = 1, 2, 3 \tag{3.5}$$

We introduce the special notation

$$|\mathbf{v}|_{l_1} = \sum_{i=1}^{3} |\upsilon^i|$$

for l_1, the norms in $\mathbb{R}^3$. By virtue of the equivalence of the norms in finite-dimensional spaces, we conclude that a number c_γ, $0 < c_\gamma < \infty$ exists such that the inequality

$$|\mathbf{v}|_{l_1} < c_\gamma |\mathbf{v}|_\gamma \tag{3.6}$$

holds for any $\mathbf{v} \in \mathbb{R}^3$.

Initially, suppose $p_1 = 1$. Then, from condition (3.4), we have

$$\int_{[0, t_n]} |\mathbf{u}_n(t)|_\gamma d\mu \to 0 \quad \text{as} \quad n \to \infty$$

On the other hand, since $\mathbf{v} \neq 0$, we obtain the chain of inequalities

$$\begin{aligned} 0 < |\mathbf{v}|_{l_1} &= \sum_{i=1}^{3} |\mathbf{v}^i| \leq \sum_{i=1}^{3} \int_{[0, t_n]} |u_n^i(t)| d\mu = \int_{[0, t_n]} \sum_{i=1}^{3} |u_n^i(t)| d\mu = \\ &= \int_{[0, t_n]} |\mathbf{u}_n(t)|_{l_1} d\mu < c_\gamma \int_{[0, t_n]} |\mathbf{u}_n(t)|_\gamma d\mu \end{aligned} \tag{3.7}$$

which leads to a contradiction.

Now, suppose $p_1 > 1$. From condition (3.4), we obtain

$$\left(\int_{[0,\,t_n]} |\mathbf{u}_n(t)|_\gamma^{p_1} d\mu\right)^{1/p_1} \to 0 \quad \text{as} \quad n \to \infty$$

We use the Hölder inequality

$$\int_{[0,\,t_n]} |\mathbf{u}_n(t)|_\gamma d\mu \le \left(\int_{[0,\,t_n]} |\mathbf{u}_n(t)|_\gamma^{p_1} d\mu\right)^{1/p_1} t_n^{1-1/p_1}$$

and, therefore,

$$t_n^{1/p_1 - 1} c_\gamma \int_{[0,\,t_n]} |\mathbf{u}_n(t)|_\gamma d\mu \to 0 \quad \text{as} \quad n \to \infty$$

Bearing in mind inequality (3.7), we conclude that $t_n^{-1} \to 0$ when $n \to \infty$. This, in turn, means that the numerical sequence $\{t_n\}$ is unbounded. Hence, for an arbitrary $\varepsilon > 0$, we choose the corresponding minimizing sequence of the processes

$$\{(t_{n_k}, \mathbf{u}_{n_k}(t), \boldsymbol{\omega}_{n_k}(t);\, t \in [0, t_{n_k}])\} \subset \mathcal{Y}(\mathbf{v}), \quad k = 1, 2 \ldots$$

which possesses the property $t_{n_k} \ge \varepsilon$ for each natural k. From equality (3.2), we now obtain

$$\int_{[0,\,\varepsilon]} |\mathbf{u}_{n_k}(t)|_\gamma^{p_1} d\mu \to 0, \quad \int_{[0,\,\varepsilon]} |\boldsymbol{\omega}_{n_k}(t)|_\sigma^{r_1} d\mu \to 0 \quad \text{as} \quad k \to \infty \tag{3.8}$$

From formulae (3.3) and (3.4), it follows that

$$|\boldsymbol{\omega}_{n_k}(t) - \mathbf{v}|_{l_1} \le \int_{[0,\,t]} |\mathbf{u}_{n_k}(s)|_{l_1} d\mu < c_\gamma \int_{[0,\,t]} |\mathbf{u}_{n_k}(s)|_\gamma d\mu$$

for each $t \in [0, \varepsilon]$. We now use the Hölder inequality and condition (3.8). We get

$$\int_{[0,\,t]} |\mathbf{u}_{n_k}(s)|_\gamma d\mu \le t^{1-1/p_1}\left(\int_{[0,\,t]} |\mathbf{u}_{n_k}(s)|_\gamma^{p_1} d\mu\right)^{1/p_1} \le \varepsilon^{1-1/p_1}\left(\int_{[0,\,\varepsilon]} |\mathbf{u}_{n_k}(s)|_\gamma^{p_1} d\mu\right)^{1/p_1} \to 0 \quad \text{as} \quad k \to \infty$$

This, in its turn, means that $|\boldsymbol{\omega}_{n_k}(t) - \mathbf{v}|_{l_1} \to 0$ when $k \to \infty$ for all $t \in [0, \varepsilon]$. But, since all the norms are equivalent in $\mathbb{R}^3$, then $|\boldsymbol{\omega}_{n_k}(t) - \mathbf{v}|_\gamma \to 0$, whence

$$|\boldsymbol{\omega}_{n_k}(t)|_\gamma \to |\mathbf{v}|_\gamma \quad \text{as} \quad k \to \infty, \quad t \in [0, \varepsilon]$$

For the sequence of non-negative functions $\{|\boldsymbol{\omega}_{n_k}(t)|_\gamma\}_{k \ge 1}$, we make use of Fatou's theorem [5, Theorem 16.2]. Then

$$\int_{[0,\,\varepsilon]} |\mathbf{v}|_\sigma d\mu \le \liminf_{k \to \infty} \int_{[0,\,\varepsilon]} |\boldsymbol{\omega}_{n_k}(s)|_\sigma d\mu \tag{3.9}$$

Using the Hölder inequality and condition (3.8) again, we obtain

$$\int_{[0,\,\varepsilon]} |\boldsymbol{\omega}_{n_k}(s)|_\sigma d\mu \le \varepsilon^{1-1/r_1}\left(\int_{[0,\,\varepsilon]} |\boldsymbol{\omega}_{n_k}(s)|_\gamma^{r_1} d\mu\right)^{1/r_1} \to 0 \quad \text{as} \quad k \to \infty$$

But it then follows from inequality (3.9) that

$$0 < \varepsilon |\mathbf{v}|_\sigma = \int_{[0,\,\varepsilon]} |\mathbf{v}|_\sigma d\mu \le 0$$

The resulting contradiction proves what is required.

4. EXISTENCE AND UNIQUENESS OF THE SOLUTION IN THE RETARDATION PROBLEM

We will now consider the non-trivial case $\mathbf{v} \neq 0$ and initially establish a property of problem (2.1), (2.2). Suppose

$$(T, \mathbf{u}(t), \omega(t); t \in [0, T]) \in \mathcal{Y}(\mathbf{v})$$

For $\alpha > 0$, we define the family of processes

$$(T_\alpha, \mathbf{u}_\alpha(t), \boldsymbol{\omega}_\alpha(t); t \in [0, T_\alpha])$$

using the formulae

$$T_\alpha = \alpha^{-1}T, \quad \mathbf{u}_\alpha(t) = \alpha\mathbf{u}(\alpha t) \quad \text{almost everywhere} \quad \boldsymbol{\omega}_\alpha(t) = \boldsymbol{\omega}(\alpha t), \quad t \in [0, T_\alpha] \tag{4.1}$$

Since

$$J(T_\alpha, \mathbf{u}_\alpha, \boldsymbol{\omega}_\alpha) = \alpha^{\frac{p_2}{p_1}(p_1 - 1)} a\|\mathbf{u}\|^{p_2}_{L^3_{p_1}([0, T]), \gamma} + \alpha^{-\frac{r_2}{r_1}} b\|\boldsymbol{\omega}\|^{r_2}_{L^3_{r_1}([0, T]), \sigma} \tag{4.2}$$

direct verification proves the inclusion

$$(T_\alpha, \mathbf{u}_\alpha(t), \boldsymbol{\omega}_\alpha(t); t \in [0, T_\alpha]) \in \mathcal{Y}(\mathbf{v}) \quad \text{for all} \quad \alpha > 0$$

For

$$a > 0, \quad b > 0 \text{ and } p_1 = 1, \quad p_2 > 0, \quad 1 \le r_1 < \infty, \quad r_2 > 0$$

problem (2.1), (2.2) is not solvable for measurable controls.

We will assume that the opposite is true. Suppose the process

$$(T^*, \mathbf{u}^*(t), \boldsymbol{\omega}^*(t); t \in [0, T^*]) \in \mathcal{Y}(\mathbf{v})$$

is the solution of the problem. Therefore,

$$J(T^*, \mathbf{u}^*, \boldsymbol{\omega}^*) = \inf_{\mathcal{Y}(\mathbf{v})} J(T, \mathbf{u}, \boldsymbol{\omega})$$

For this process, using formulae (4.1), we construct the family of processes

$$(T^*_\alpha, \mathbf{u}^*_\alpha(t), \boldsymbol{\omega}^*_\alpha(t); t \in [0, T^*_\alpha]) \in \mathcal{Y}(\mathbf{v})$$

which are admissible in the same problem and conclude from equality (4.2) that, when $\alpha \to \infty$

$$J(T^*_\alpha, \mathbf{u}^*_\alpha, \boldsymbol{\omega}^*_\alpha) = a\|\mathbf{u}^*\|^{p_2}_{L^3_{p_1}([0, T^*]), \gamma} + \alpha^{-\frac{r_2}{r_1}} b\|\boldsymbol{\omega}^*\|^{r_2}_{L^3_{r_1}([0, T^*]), \sigma} \to$$

$$\to a\|\mathbf{u}^*\|^{p_2}_{L^3_{p_1}([0, T^*]), \gamma} < a\|\mathbf{u}^*\|^{p_2}_{L^3_{p_1}([0, T^*]), \gamma} + b\|\boldsymbol{\omega}^*\|^{r_2}_{L^3_{r_1}([0, T^*]), \sigma} = \inf_{\mathcal{Y}(\mathbf{v})} J(T, \mathbf{u}, \boldsymbol{\omega})$$

The inequality holds since inequality (3.2) implies that

$$\|\boldsymbol{\omega}^*\|_{L^3_{r_1}([0, T^*]), \sigma} > 0$$

The resulting contradiction proves what is required.

Now suppose that $p_1 > 1$ and assume that problem (2.1), (2.2) has at least one solution

$$(T^*, \mathbf{u}^*(t), \boldsymbol{\omega}^*(t); t \in [0, T^*]) \in \mathcal{Y}(\mathbf{v})$$

On constructing, according to this process, the family of admissible processes using formulae (4.1), for the corresponding values of the functionals we get the function

$$f(\alpha) = J(T^*_\alpha, \mathbf{u}^*_\alpha, \boldsymbol{\omega}^*_\alpha) = \alpha^{\frac{p_2}{p_1}(p_1-1)} a\|\mathbf{u}^*\|^{p_2}_{L^3_{p_1}([0,T^*]),\gamma} + \alpha^{-\frac{r_2}{r_1}} b\|\boldsymbol{\omega}^*(t)\|^{r_2}_{L^3_{r_1}([0,T^*]),\sigma}$$

which has a unique global minimum at the point

$$\alpha^* = \left\{\frac{r_2}{r_1}\frac{1}{p_1-1}\frac{p_1 b}{p_2 a}\frac{\|\boldsymbol{\omega}^*\|^{r_2}_{L^3_{r_1}([0,T^*]),\sigma}}{\|\mathbf{u}^*\|^{p_2}_{L^3_{p_1}([0,T^*]),\gamma}}\right\}^s, \quad s = \left(\frac{p_2}{p_1}(p_1-1)+\frac{r_2}{r_1}\right)^{-1}$$

Since

$$f(1) = \inf_{\mathcal{Y}(\mathbf{v})} J(T, \mathbf{u}, \boldsymbol{\omega})$$

by construction, then, from the system of inequalities

$$f(\alpha^*) \le f(\alpha), \quad \alpha > 0; \quad f(1) \le f(\alpha^*)$$

and the fact that there is only a single extremum of $f(\alpha)$, we conclude that

$$\alpha^* = 1$$

whence

$$\frac{r_2}{r_1} b\|\boldsymbol{\omega}^*\|^{r_2}_{L^3_{r_1}([0,T^*]),\sigma} = \frac{p_2}{p_1}(p_1-1)a\|\mathbf{u}^*\|^{p_2}_{L^3_{p_1}([0,T^*]),\gamma}$$

This, in turn, means that the following representation holds

$$\inf_{\mathcal{Y}(\mathbf{v})} J(T, \mathbf{u}, \boldsymbol{\omega}) = J(T^*, \mathbf{u}^*, \boldsymbol{\omega}^*) = \left\{1 + \frac{r_1 p_2}{r_2 p_1}(p_1-1)\right\} a\|\mathbf{u}^*\|^{p_2}_{L^3_{p_1}([0,T^*]),\gamma} = \tag{4.3}$$
$$= \left\{1 + \frac{r_2}{r_1}\frac{1}{p_1-1}\frac{p_1}{p_2}\right\} b\|\boldsymbol{\omega}^*\|^{r_2}_{L^3_{r_1}([0,T^*]),\sigma}$$

We will now assume that at least one of the normalized spaces $(\mathbb{R}^3, |\cdot|_\gamma)$ or $(\mathbb{R}^3, |\cdot|_\sigma)$ is rigorously normalized. In this case, the following assumption holds for all possible values (1.4) of the parameters: if problem (2.1), (2.2) has a solution from $\mathcal{Y}(\mathbf{v})$, then it is unique (apart from equivalent controls).

We shall argue from the opposite and assume that there are two permissible optimal processes. Without loss of generality, it can be assumed that the time of the completion of a manoeuvre is the same for these processes. This can always be achieved by extending one of the processes in a trivial manner (with a null control) since a state of rest is the final position. Thus, suppose that

$$(T^*, \mathbf{u}^*_i(t), \boldsymbol{\omega}^*_i(t); t \in [0, T^*]) \in \mathcal{Y}(\mathbf{v}), \quad i = 1, 2$$

and that the following equalities hold

$$J(T^*, \mathbf{u}^*_i, \boldsymbol{\omega}^*_i) = \inf_{\mathcal{Y}(\mathbf{v})} J(T, \mathbf{u}, \boldsymbol{\omega})$$

Using equality (4.3), we then arrive at the conclusion that the following equalities also hold

$$\|\mathbf{u}^*_1\|_{L^3_{p_1}([0,T^*]),\gamma} = \|\mathbf{u}^*_2\|_{L^3_{p_1}([0,T^*]),\gamma}, \quad \|\boldsymbol{\omega}^*_1\|_{L^3_{r_1}([0,T^*]),\sigma} = \|\boldsymbol{\omega}^*_2\|_{L^3_{r_1}([0,T^*]),\sigma} \tag{4.4}$$

We now introduce into consideration the set $\mathcal{A} = \{t \in [0, T^*]$: the functions $\mathbf{u}^*_1(t)$ and $\mathbf{u}^*_2(t)$ are not positively proportional$\}$. The set $\mathcal{A}$ is obviously measurable, and we assume that $\mu(\mathcal{A}) > 0$. We will now show that this is impossible.

To be specific, we will assume that the space $(\mathbb{R}^3, |\cdot|_\gamma)$ is a strictly normalized space. In this case, for any $\lambda \in [0, 1]$, we have

$$|\lambda \mathbf{u}_1^*(t) + (1-\lambda)\mathbf{u}_2^*(t)|_\gamma < \lambda|\mathbf{u}_1^*(t)|_\gamma + (1-\lambda)|\mathbf{u}_2^*(t)|_\gamma, \quad t \in \mathcal{A}$$

The following chain of inequalities

$$\|\lambda \mathbf{u}_1^* + (1-\lambda)\mathbf{u}_2^*\|^{p_1}_{L^3_{p_1}([0,T^*]),\gamma} = \||\lambda \mathbf{u}_2^* + (1-\lambda)\mathbf{u}_2^*|_\gamma\|^{p_1}_{L_{p_1}([0,T^*])} =$$

$$= \int_{[0,T^*]} |\lambda \mathbf{u}_1^*(t) + (1-\lambda)\mathbf{u}_2^*(t)|_\gamma^{p_1} d\mu =$$

$$= \int_{\mathcal{A}} |\lambda \mathbf{u}_1^*(t) + (1-\lambda)\mathbf{u}_2^*(t)|_\gamma^{p_1} d\mu + \int_{[0,T^*]\backslash\mathcal{A}} |\lambda \mathbf{u}_1^*(t) + (1-\lambda)\mathbf{u}_2^*(t)|_\gamma^{p_1} d\mu <$$

$$< \int_{\mathcal{A}} (\lambda|\mathbf{u}_1^*(t)|_\gamma + (1-\lambda)|\mathbf{u}_2^*(t)|_\gamma)^{p_1} d\mu + \int_{[0,T^*]\backslash\mathcal{A}} (\lambda|\mathbf{u}_1^*(t)|_\gamma + (1-\lambda)|\mathbf{u}_2^*(t)|_\gamma)^{p_1} d\mu =$$

$$= \int_{[0,T^*]} (\lambda|\mathbf{u}_1^*(t)|_\gamma + (1-\lambda)|\mathbf{u}_2^*(t)|_\gamma)^{p_1} d\mu = \|\lambda|\mathbf{u}_1^*|_\gamma + (1-\lambda)|\mathbf{u}_2^*|_\gamma\|^{p_1}_{L_{p_1}([0,T^*])}$$

leads to the conclusion that

$$\|\lambda \mathbf{u}_1^* + (1-\lambda)\mathbf{u}_2^*\|_{L^3_{p_1}([0,T^*]),\gamma} < \|\lambda|\mathbf{u}_1^*|_\gamma + (1-\lambda)|\mathbf{u}_2^*|_\gamma\|_{L_{p_1}([0,T^*])} \le$$

$$\le \lambda\||\mathbf{u}_1^*|_\gamma\|_{L_{p_1}([0,T^*])} + (1-\lambda)\||\mathbf{u}_2^*|_\gamma\|_{L_{p_1}([0,T^*])} =$$

$$= \lambda\|\mathbf{u}_1^*\|_{L^3_{p_1}([0,T^*]),\gamma} + (1-\lambda)\|\mathbf{u}_2^*\|_{L^3_{p_1}([0,T^*]),\gamma} = \|\mathbf{u}_1^*\|_{L^3_{p_1}([0,T^*]),\gamma}$$

The equality (4.4) has been used here.

For an arbitrary normalized space $(\mathbb{R}^3, |\cdot|_\sigma)$, from equality (4.4) we obtain the non-rigorous inequality

$$\|\lambda \boldsymbol{\omega}_1^* + (1-\lambda)\boldsymbol{\omega}_2^*\|_{L^3_{r_1}([0,T^*]),\sigma} \le \|\boldsymbol{\omega}_2^*\|_{L^3_{r_1}([0,T^*]),\sigma}$$

Direct verification shows that a convex combination of processes from $\mathcal{Y}(\mathbf{v})$ with the same completion time again generates an admissible process from $\mathcal{Y}(\mathbf{v})$. The inclusion

$$(T^*, \lambda \mathbf{u}_1^*(t) + (1-\lambda)\mathbf{u}_2^*(t), \lambda \boldsymbol{\omega}_1^*(t) + (1-\lambda)\boldsymbol{\omega}_2^*(t);\, t \in [0,T^*]) \in \mathcal{Y}(\mathbf{v})$$

therefore holds for any $\lambda \in [0, 1]$.

However, if $\mu(\mathcal{A}) > 0$, we obtain the inequality

$$J(T^*, \lambda \mathbf{u}_1^* + (1-\lambda)\mathbf{u}_2^*, \lambda \boldsymbol{\omega}_1^* + (1-\lambda)\boldsymbol{\omega}_2^*) = \tag{4.5}$$

$$= a\|\lambda \mathbf{u}_1^* + (1-\lambda)\mathbf{u}_2^*\|^{p_2}_{L^3_{p_1}([0,T^*]),\gamma} + b\|\lambda \boldsymbol{\omega}_1^* + (1-\lambda)\boldsymbol{\omega}_2^*\|^{r_2}_{L^3_{r_1}([0,T^*]),\sigma} <$$

$$< a\|\mathbf{u}_1^*\|^{p_2}_{L^3_{p_1}([0,T^*]),\gamma} + b\|\boldsymbol{\omega}_1^*\|^{r_2}_{L^3_{r_1}([0,T^*]),\sigma} = J(T^*, \mathbf{u}_1^*, \boldsymbol{\omega}_1^*) = \inf_{\mathcal{Y}(\mathbf{v})} J(T, \mathbf{u}, \boldsymbol{\omega})$$

for the values of the functional, which is impossible. The resulting contradiction shows that $\mu(\mathcal{A}) = 0$ and, consequently, a positive function $x(t)$ exists almost everywhere in $[0, T^*]$ for which the following equality holds

$$\mathbf{u}_1^*(t) = x(t)\mathbf{u}_2^*(t) \quad \text{for almost all} \quad t \in [0, T^*] \tag{4.6}$$

We will now show that $x(t) = 1$ almost everywhere in $t \in [0, T^*]$. It follows from the preceding discussion that, when $p_1 = 1$, the problem does not have admissible solutions. We will therefore assume that $1 < p_1 < \infty$. However, it is well known [6, p. 594]; [7, p. 254] that the space $L_{p_1}([0, T^*])$ is strictly normalized. Hence, if the equality

$$|\mathbf{u}_1^*(t)|_\gamma = \beta|\mathbf{u}_2^*(t)|_\gamma \quad \text{for almost all} \quad t \in [0, T^*]$$

is not satisfied for some number $\beta > 0$, then, on taking account of equality (4.4)

$$\|\lambda \mathbf{u}_1^* + (1-\lambda)\mathbf{u}_2^*\|_{L^3_{p_1}([0,T^*]),\gamma} = \|\lambda|\mathbf{u}_1^*|_\gamma + (1-\lambda)|\mathbf{u}_2^*|_\gamma\|_{L_{p_1}([0,T^*])} <$$

$$< \lambda\||\mathbf{u}_1^*|_\gamma\|_{L_{p_1}([0,T^*])} + (1-\lambda)\||\mathbf{u}_2^*|_\gamma\|_{L_{p_1}([0,T^*])} = \|\mathbf{u}_1^*\|_{L^3_{p_1}([0,T^*])}$$

we arrive at a contradiction similar to inequality (4.5). Consequently, for a certain $\beta > 0$, we have

$$|\mathbf{u}_1^*(t)|_\gamma = \beta|\mathbf{u}_2^*(t)|_\gamma \quad \text{for almost all} \quad t \in [0, T^*] \tag{4.7}$$

The conclusion that

$$x(t) = \beta = 1 \quad \text{for almost all} \quad t \in [0, T^*]$$

follows from a comparison of formulae (4.7), (4.6) and (4.4).

Hence, the uniqueness of the solution of problem (2.1), (2.2) is proved in this case.

Since, according to assumption (1.4), we have $r_1 > 1$, then the reasoning in the case of a strictly normalized space $(\mathbb{R}^3, |\cdot|_\gamma)$ is carried out in a similar manner.

5. THE EXISTENCE OF SOLUTIONS IN THE PROBLEM OF REORIENTATION

The purpose of this section is to demonstrate the fact that the problem of optimal orientation (1.1)–(1.3), under the assumptions used, almost never has a solution in the class of measurable controls. We will now refine this assertion.

We shall assume that inequalities (1.4) establish a possible range of values of the parameters of the functional and that one of the spaces $(\mathbb{R}^3, |\cdot|_\gamma)$ or $(\mathbb{R}^3, |\cdot|_\sigma)$ is strictly normalized.

Suppose $\mathcal{V} \subset \mathbb{R}^3$ is the set of those initial angular velocities $\mathbf{v}$ for which the corresponding problem (2.1), (2.2) of optimal retardation has an admissible solution (from $\mathcal{Y}(\mathbf{v})$). In other words, if $\mathbf{v} \in \mathcal{V}$, then there is a unique process

$$(T_\upsilon, \mathbf{u}_\upsilon(t), \boldsymbol{\omega}_\upsilon(t); t \in [0, T_\upsilon]) \in \mathcal{Y}(\mathbf{v})$$

for which the following equality holds

$$J(T_\upsilon, \mathbf{u}_\upsilon, \boldsymbol{\omega}_\upsilon) = \inf_{\mathcal{Y}(\mathbf{v})} J(T, \mathbf{u}, \boldsymbol{\omega})$$

A unique matrix $D_\upsilon = D(T_\upsilon) \in SO(3)$, which is a solution of the Cauchy problem

$$D(0) = I_3, \quad \dot{D} = -S(\boldsymbol{\omega}_\upsilon)D, \quad t \in [0, T_\upsilon]$$

therefore corresponds to each $\mathbf{v} \in \mathcal{V}$.

Suppose $B, C \in SO(3)$ and $\mathbf{v} \in \mathbb{R}^3$ are the matrix and the vector which determine the boundary conditions in problem (1.1)–(1.3). If $\mathbf{v} \in \mathcal{V}$ and $CB^T = D_\upsilon$, then, by virtue of equality (2.8), a solution of problem (1.1)–(1.3) is also a solution of problems (2.1), (2.2).

If either $\mathbf{v} \notin \mathcal{V}$ or $CB^T \neq D_\upsilon$, then the problem of optimal reorientation (1.1)–(1.3) with a free completion time is unsolvable for measurable controls.

Actually, suppose $\mathbf{v} \notin \mathcal{V}$. We will assume that there is an optimal process

$$(T^*, \mathbf{u}^*(t), \boldsymbol{\omega}^*(t), A^*(t); t \in [0, T^*]) \in \mathcal{X}(B, C, \mathbf{v})$$

in the case of the boundary conditions $B, C \in SO(3)$ and $\mathbf{v} \in \mathbb{R}^3$ in problem (1.1)–(1.3).

But, by virtue of equality (2.8), the process

$$(T^*, \mathbf{u}^*(t), \boldsymbol{\omega}^*(t); t \in [0, T^*]) \in \mathcal{Y}(\mathbf{v})$$

is then a solution of problem (2.1), (2.2). That is a contradiction.

Now, suppose $\mathbf{v} \notin \mathcal{V}$ and $CB^T \neq D_\upsilon$. Again, if

$$(T^*, \mathbf{u}^*(t), \boldsymbol{\omega}^*(t), A^*(t);\ t \in [0, T^*]) \in \mathcal{X}(B, C, \mathbf{v})$$

is a solution of problem (1.1)–(1.3), then

$$(T^*, \mathbf{u}^*(t), \boldsymbol{\omega}^*(t);\ t \in [0, T^*]) \in \mathcal{Y}(\mathbf{v})$$

is a solution of problem (2.1), (2.2). But, since $CB^T \neq D_\upsilon$, there are then two different solutions in the case of the optimal retardation problem, which is impossible under the assumptions which have been used.

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